THE TWENTY-FIFTH SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE twenty-fifth summer meeting of the Society was held, by invitation, at Dartmouth College on Wednesday, Thursday and Friday, September 4–6, 1918, connecting with the meeting of the Mathematical Association, which began on Friday morning. The joint dinner of the two organizations, on Thursday evening, was attended by fifty-six members and friends, who were greeted by Dean Laycock in the name of the College. At the joint session on Friday morning Professor A. G. Webster gave an address on "Mathematics of warfare."

The college dormitories were opened for the accommodation of the visitors, and meals were served in the commons. Headquarters and general gathering place between the sessions was provided in College Hall, where an informal reception was held on Wednesday evening. A letter of welcome from Business Director Keyes tendered the hospitality of the College to the visiting societies. Excursions into the country about Hanover were arranged for the closing days of the meetings. At the joint session a vote of thanks was extended to the college authorities for their generous cooperation toward a successful occasion.

The meeting included an evening session on Wednesday and the usual morning and afternoon sessions on Thursday, as well as the joint session on Friday morning. The attendance included the following forty-six members of the Society:

Professor H. L. Agard, Professor R. C. Archibald, Professor R. D. Beetle, Professor G. D. Birkhoff, Professor Daniel Buchanan, Professor W. D. Cairns, Professor W. B. Carver, Professor F. N. Cole, Professor Julia T. Colpitts, Dr. Lennie P. Copeland, Professor Louise D. Cummings, Dr. Mary F. Curtis, Professor C. H. Currier, Professor E. L. Dodd, Mr. T. C. Frye, Professor A. S. Gale, Professor O. E. Glenn, Mr. B. F. Groat, Professor C. F. Gummer, Professor J. G. Hardy, Professor C. N. Haskins, Professor E. V. Huntington, Professor W. W. Johnson, Professor Florence P. Lewis, Professor Helen A. Merrill, Dr. A. L. Miller, Professor F. M. Morgan, Professor G. D. Olds, Professor F. W. Owens, Professor Anna H. Palmié, Professor A. D. Pitcher, Professor J. M. Poor, Mr.

L. H. Rice, Professor R. G. D. Richardson, Professor E. D. Roe, Jr., Dr. Josephine R. Roe, Professor Clara E. Smith, Professor Sarah E. Smith, Professor H. W. Tyler, Professor Oswald Veblen, Professor C. A. Waldo, Professor A. G. Webster, Mr. J. K. Whittemore, Professor C. B. Williams, Professor F. N. Willson, Professor J. W. Young.

Professor W. W. Johnson presided at the opening session on Wednesday evening, Professor H. W. Tyler at the sessions on Thursday, and Professor G. D. Birkhoff at the joint session on Friday morning. The Council announced the election of the following persons to membership in the Society: Professor A. L. Candy, University of Nebraska; Mr. J. R. Carson, American Telephone and Telegraph Company; Mr. R. S. Hoyt, American Telephone and Telegraph Company; Dr. K. W. Lamson, Columbia University; Professor A. S. Merrill, University of Montana; Mr. F. H. Murray, Harvard University; Mr. H. W. Nichols, Western Electric Company; Professor W. E. Patten, Government Institute of Technology, Shanghai, China. Nine applications for membership in the Society were received.

The following papers were read at this meeting:

(1) Dr. L. B. ROBINSON: "A curious system of polynomials."

(2) Professor G. A. MILLER: "Groups generated by two operators whose relative transforms are equal to each other."

(3) Professor P. J. DANIELL: "Differentiation with respect to a function of limited variation."

(4) Mr. B. F. GROAT: "Models and hydraulic similarity."

(5) Professor L. C. MATHEWSON: "On the groups of isomorphisms of a system of abelian groups of order p^m and type $(n, 1, 1, \ldots, 1)$."

(6) Mr. C. N. REYNOLDS: "On the zeros of solutions of linear differential equations of the fourth order."

(7) Professor J. E. ROWE: "Related invariants of two rational sextics."

(8) Professor W. W. JOHNSON: "The nature and history of Napier's rules of circular parts."

(9) Professor O. E. GLENN: "On a new treatment of theorems of finiteness."

(10) Professor LOUISE D. CUMMINGS: "The trains for the 36 groupless triad systems on 15 elements."

(11) Dr. JOSEPHINE R. ROE: "Interfunctional expressibility problems of symmetric functions (third paper)."

(12) Mr. B. F. GROAT: "Equations of the elastic catenary."

(13) Professor C. H. FORSYTH: "Relative distributions."

(14) Professor W. D. CAIRNS: "A derivation of the equation of the normal probability curve."

(15) Dr. MARY F. CURTIS: "Curves invariant under point transformations of special type."

(16) Professor G. D. BIRKHOFF: "On stability in dynamics."

(17) Professor DANIEL BUCHANAN: "Periodic orbits on a surface of revolution."

(18) Professor C. N. HASKINS: "Note on the roots of the function $P(x)$ associated with the gamma function" (preliminary communication).

(19) Dr. A. R. SCHWEITZER: "On the iterative properties of an abstract group (third paper)."

(20) Mrs. CHRISTINE LADD-FRANKLIN: "Bertrand Russell and symbol logic."

Mrs. Ladd-Franklin was introduced by Professor Fiske. In the absence of the authors the papers of Professor Miller, Professor Daniell, Professor Mathewson, Mr. Reynolds, Professor Rowe, Professor Forsyth, and Dr. Schweitzer were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Given the system of equations

$$p_1 m_{11}^2 + 2p_2 m_{11} m_{21} + 2p_3 m_{11} m_{31} + p_4 m_{21}^2 + 2p_5 m_{21} m_{31} + p_6 m_{31}^2 = 0,$$

$$p_1 m_{11} m_{12} + p_2(m_{11} m_{22} + m_{12} m_{21}) + p_3(m_{11} m_{32} + m_{12} m_{31}) + p_4 m_{21} m_{22} + p_5(m_{21} m_{32} + m_{22} m_{31}) + p_6 m_{31} m_{32} = 0,$$

$$p_1 m_{12}^2 + 2p_2 m_{12} m_{22} + 2p_3 m_{12} m_{32} + p_4 m_{22}^2 + 2p_5 m_{22} m_{32} + p_6 m_{32}^2 = 0 \qquad (p_1 \neq 0);$$

to this system corresponds a determinant $|m_{ij}|$ $(i, j = 1, 2, 3)$. Dr. Robinson has demonstrated that, unless the relation

$$(p_2^2 - p_1 p_4)(p_3^2 - p_1 p_6) - (p_2 p_3 - p_1 p_5)^2 = 0$$

exists, the solutions of the given system of polynomials annul all the determinants of the matrix

$$\begin{Vmatrix} m_{11} & m_{21} & m_{31} \\ m_{12} & m_{22} & m_{32} \end{Vmatrix}$$

An analogous theorem holds for systems of polynomials which correspond to the determinant $|m_{ij}|$ $(i, j = 1, 2, \ldots, n)$.

2. When two operators s and t of a group satisfy the condition $s^{-1}ts = t^{-1}st$, they are of the same order α. If they are not otherwise restricted, Professor Miller shows that they generate a group of order $\alpha(2^\alpha - 1)$ which has a cyclic commutator subgroup of order $2^\alpha - 1$, and is generated by this subgroup and an operator of order α which transforms each of its operators into its square. There is one and only one such group for every positive integral value of α and this infinite system may be called the equitransform system of groups. When $\alpha = 2$ there results the symmetric group of order 6; when $\alpha = 3$, the semi-metacyclic group of order 21; etc. Each group of the system contains a set of $2^\alpha - 1$ conjugate cyclic subgroups of order α and all its operators of order 2 are conjugate whenever such operators occur.

3. In this paper Professor Daniell defines the derivative with respect to a function of limited variation, proves a modification of Vitali's theorem, and finally proves that, under certain restrictions, the original function is the integral of its derivative. A law for integration by parts is found and it is proved that any Stieltjes integral can be expressed as a single Stieltjes integral with respect to a non-decreasing function. The paper is to appear in the *Transactions*.

4. In this paper, Mr. Groat discusses the proper scale ratios to employ to secure similarity of flow of fluids in geometrically similar channels. The determinations are related to theory developed by him in a paper soon to appear in the *Transactions of the American Society of Civil Engineers*, volume 82.

It is shown that similarity of flow requires a scale ratio $L = N^{2/3} \div G^{1/3}$, where L = scale ratio, N = ratio of corresponding kinematic viscosities, and G = ratio of homologous accelerations,—all dimensionless ratios. The similarity then extends to details of the flow, including forces due to viscosity.

Attention is called to the fact that in ordinary model experi-

ments G is unity. Therefore $L = N^{2/3}$. This shows that perfect similarity requires different values of the kinematic viscosity, which in turn requires either different fluids or different viscosity conditions of the same fluid in model and prototype.

Lamb (Hydrodynamics, page 537) states that the kinematic viscosity of air varies inversely as the pressure. If this is true, the sizes of model aeroplane propellers, for example, may be determined for various conditions as follows:

Pressure in atmospheres	1000	64	27
Value of N	1/1000	1/64	1/27
Scale ratio of model	1/100	1/16	1/9

This assumes, of course, that velocities do not approach too closely the velocity of sound in the air or other fluid involved.

The same effect might be secured by employing another fluid, as mercury, for example. The following table gives approximate ratios in two cases for different conditions: (*a*) model in water of prototype in air; (*b*) model in mercury of prototype in water.

		Temperature of water (C.)		
Temperature (C.)		10°	15°	25°
10°	Air	.204	.186	.158
	Mercury		.219	.255
20°	Air	.190	.174	.148
	Mercury		.220	.257
40°	Mercury		.210	.246

It is of importance to note that the scale ratio of a model in mercury of prototype in air can be obtained by taking the product of any two ratios for an intermediate field. Thus, the scale ratio for air at 20° and mercury at 40° can be secured by taking the product of the air-water and water-mercury ratios for water at any given temperature, say 15°, thus: $L = .174 \times .210 = .036$. Therefore a model aeroplane propeller of about 1/27 full size can be tested in mercury at 40° C., the air being at 20° C., with caution as to the velocity of sound in the media.

5. Moore has shown that the group of isomorphisms of an abelian group of order p^m, and type $(1, 1, \ldots, 1)$ is the linear congruence group,* Miller has considered the automorphisms

* *Cf.* also Burnside, Theory of Groups (1897), §§ 171, 172 and Chap. XIV.

of an abelian group of order p^m, type $(m-1, 1)$,* and Ranum has developed the group of isomorphisms of any given abelian group of order p^m through a consideration of the group of classes of congruent matrices.† In Professor Mathewson's paper the viewpoint is different and the groups are treated wholly as abstract groups. The object is to study the groups of isomorphisms of the system of abelian groups of order p^m and type $(n, 1, 1, \ldots, 1)$, $n > 1$, and to show that these groups of isomorphisms may be built upon the group of isomorphisms of an abelian group of order p^m and type $(1, 1, \ldots, 1)$ or that the group of isomorphisms of the system under study ($n = 2, 3, 4, \ldots$) is an extension of the one of the system just before it and of index p in it.

6. Professor Birkhoff has proven theorems of oscillation and comparison for the solutions of ordinary linear differential equations of the third order, which are analogous to Sturm's well-known theorems concerning the solutions of linear differential equations of the second order (*Annals of Mathematics*, volume 12 (1911)). In the present paper Mr. Reynolds has proven similar theorems concerning the solutions of self-adjoint linear differential equations of the fourth order, and an oscillation theorem for the general linear differential equation of the fourth order.

7. Professor Rowe's paper appeared in the October BULLETIN.

8. In this paper Professor Johnson gives an historical survey of Napier's rules of the circular parts of right-angled spherical triangles. The principle underlying the rules is examined, and its partial anticipation as found in the early history of trigonometry. Some other cases of circular parts are also discussed.

9. An abstract of Professor Glenn's paper appeared in the March (1918) number of the BULLETIN, in connection with a preliminary report upon the subject, presented by title at the December meeting of the Society. Material since added includes a determination of the complete systems of orthog-

* Miller, *Transactions Amer. Math. Society*, vol. 2 (1901), pp. 259–264.
† Ranum, *Transactions Amer. Math. Society*, vol. 8 (1907), pp. 71–91.

onal and of boolean invariants and covariants of the binary quintic.

10. In Professor Cummings' paper the sets of trains for the 36 groupless triad systems on fifteen elements are determined, and all types of trains which may occur in the 80 non-congruent systems Δ_{15} are exhibited.

11. In this paper, which is a continuation of those presented at meetings of the Society in October, 1916, and September, 1917, Dr. Roe formulates laws for zero coefficients in interfunctional expressibility tables of symmetric functions and proves triangularity as a property of certain of these tables.

12. Mr. Groat's second paper deals with equations of the elastic catenary deduced about the year 1902. The applicability to suspended cables was established at the time by taking transit observations upon ropes of a cable-way. Since then the theory has been expanded into a treatise adapted to practical engineering, which, though not yet published, has been employed upon transmission-line design and construction.

If Young's modulus be introduced, the following equations result:

$$u + c_0 = \theta + m \sinh \theta,$$

$$v + c_1 = \cosh \theta + \tfrac{1}{2} m \sinh^2 \theta,$$

$$w + c_2 = \sinh \theta + \tfrac{1}{2} m (\sinh \theta \cosh \theta + \theta),$$

where $u = x \div c$, $v = y \div c$, $w = z \div c$; u, v, and x, y, being similar systems of rectangular coordinates, and w, z, the corresponding rectifications of the similar curves from vertex to point u, v or x, y. The parameters are m and c, while θ is an independent variable which may be eliminated from any pair of the three equations, leaving a transcendental; for example, the rectangular equation of the elastic catenary.

Moreover, by employing in place of Young's modulus one in which the actual intensity of stress upon the current sectional area replaces that upon the original sectional area of the unstretched cable, another rectangular equation may be derived which is less cumbersome and fully as exact as the former, for all stresses below the elastic limit. There is evidence to show that such a modulus is more nearly constant in actual experiments than Young's.

With certain minor approximations the resulting equations are:

$$mv = \log[1 + m \cosh(\mu u)],$$

$$mw = u - 2 \tanh^{-1}\left[\sqrt{\frac{1-m}{1+m}} \tanh(\mu u)\right],$$

where $\mu = \sqrt{1 - m^2}$ and m and c have the same meaning as before, i. e., $m = H \div T$, $c = H \div s$, H being the total tension at the vertex, s the constant weight per unit length of unstretched cable, and T a constant depending upon the size and elastic properties involved.

Thus, it is possible to discuss any elastic catenary by either method as regards the effects of different loadings, temperatures, and other conditions, and this has been accomplished in numerous cases. Practical processes are simplified by employing different parts of the two methods in combination, the results being substantially the same by either, though obtained with greater facility by one or the other.

A general process for elastic curves of all kinds is pointed out, specifically that for an "elastic parabola."

13. In Professor Forsyth's paper a method is derived for making a series of interpolations in a systematic manner in a frequency distribution or a sequence of values separated by equal intervals to give a new frequency distribution or sequence of values separated by equal intervals which may or may not be the same as the original intervals. Thus, for illustration, given the population of a country such as the United States at the middle of the first of each year of several decennial periods, the method may be used to determine the population at the beginning of each such year.

14. The symmetrical distribution of magnitudes about their mean is commonly represented by a "polygon" whose equally spaced ordinates are proportional to the terms of the expansion of $(1 + 1)^n$. A direct proof, using Stirling's formula, is given by Professor Cairns of the theorem frequently quoted without proof in texts that, as n is increased indefinitely, the polygon (with a finite middle ordinate) approaches as its limiting form the normal curve $y = ke^{-h^2x^2}$. The method consists essentially in controlling what may be called the points of in-

flexion of the polygons so that these points approach predetermined positions on each side of the mean.

15. In his discussion of the curves defined by a differential equation of the form $dx/X = dy/Y$ Poincaré has investigated the curves invariant under the transformation

$$\tau: \quad \begin{aligned} x_1 &= S_1 x + \sum_{i+j=2}^{\infty} a_{ij} x^i y^j, \\ y_1 &= S_2 y + \sum_{i+j=2}^{\infty} b_{ij} x^i y^j, \end{aligned}$$

where $S_1 > 1 > S_2 > 0$, $S_2 > 1 > S_1 > 0$; and has shown that there always exist two invariant analytic curves through the invariant point (x_0, y_0). M. Lattes, using Poincare's methods, has shown that, when S_1, S_2 are distinct and not zero and one is not a positive integral power of the other, the same result holds. Lie's theory of infinitesimal transformations applied to the case $S_1 = S_2 = 1$—one of the cases in which Poincaré's methods fail—enables Dr. Curtis to determine the number of curves formally invariant under τ through the origin.

16. Professor Birkhoff's paper is devoted to questions concerning the stability of periodic orbits in dynamical systems with two degrees of freedom. The following theorem forms an important part of this investigation:

Given two closed curves C_1, C_2 about a point O in their plane, each cut by every radial half line from O once and only once, and given a one-to-one, direct, continuous, area-preserving transformation T of the ring C_1C_2 into itself, such that the image of any radial line under T is cut by any radial line at most once; then there exists an infinite connected set of invariant curves C about O, each cut by every radial line once and only once.

The results concerning stability will form part of a paper to appear in the *Proceedings of the American Academy of Arts and Sciences*.

17. In this paper Professor Buchanan determines the periodic orbits described by a particle which moves on a smooth surface of revolution, the axis of which is vertical. Periodic orbits are first determined for the vertical paraboloid of revo-

lution in much the same way as the well-known problem of the spherical pendulum is solved. Then the analytic continuation of these orbits is made with respect to a certain parameter which is introduced to differentiate the more general surface of revolution from the paraboloid.

18. Bourguet showed* that the function

$$P(z) \equiv \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{\nu!} \frac{1}{z+\nu}$$

has two real roots in each of the intervals $-2n-2 < x < -2n-1$, $n \geqq 3$, and that these roots are of the form $-2n-1-\xi_n$, $-2n-2+\eta_n$, where ξ_n is of the order of $1/(2n)!$. Gronwall has recently shown† that $1/(2n)! < \xi_n < 6/(2n)!$, $1/(2n+1)! < \eta_n < 6/(2n+1)!$.

Professor Haskins points out that, though Bourguet's paper contains minor inaccuracies, typographical and other, his approximation is closer than that of Gronwall from $n = 8$ onward and gives the result that $\xi_n < (3+\epsilon_n)/(2n)!$, where $\lim_{n=\infty} \epsilon_n = 0$.

Formulas given but not developed by Bourguet lead to an approximation equalling that of Gronwall for $n \geqq 2$ and with the limiting value $3/(2n)!$ for large n. Extension of Bourguet's methods leads to still closer approximation, e. g., $2.76/4! < \xi_2 < 2.91/4!$

19. Dr. Schweitzer constructs a table of undefined relations for the postulates for an abstract group, the relations being connoted by a functional notation. The first set of 2^2 undefined relations is of the second degree and has been treated previously.* The kth set of 2^{k+1} undefined relations is of the $(k+1)$st degree and is obtained by induction from the set for $k = k_0$; e. g., for $k = 3$ the indefinables are, $F_{13}(x_1x_2x_3)$, $F_{23}(x_1x_2^{-1}x_3)$, $F_{33}(x_1^{-1}x_2x_3)$, $F_{43}(x_1^{-1}x_2^{-1}x_3)$, $F_{53}(x_1x_2x_3^{-1})$, etc. Conceivably in the case of k variables one has $\Sigma_{s=1}^{m} {}^mC_s$ classes of systems of postulates for an abstract group, where the sub-

* Bourguet, L., *Comptes Rendus*, vol. 96 (1883), pp. 1307–1310; 1457–1490.

† Gronwall, T. H., *Annales Scientifiques de l'Ecole Normale Supérieure*, series (3), vol. 33 (1916), pp. 381–393.

* Compare abstracts, BULLETIN, May, 1918, p. 371, June, 1918, p. 428. In the former abstract, the relations Π_2, Π_2, Π_1, should be stated as follows: Π_2. $\phi\{f(x,y),y\} = x$, Π_3. $f\{\phi(x,y),y\} = x$, Π_1. $f\{f(y,x),f(z,x)\} = f(y,z)$.

script s denotes the number of indefinables involved in a system of postulates and $m = 2^{k+1}$.

In the present paper the author constructs for $k = 2$, five classes of systems of postulates for a group, finite or infinite, based on three and four indefinables respectively; also eight systems of postulates are constructed for $k = 3$, each system involving a single indefinable, F_{12}, . . ., F_{83} respectively. Infinitudes of new equations in iterative compositions are given (with special reference to the preceding 2^{k+1} indefinables) and their interpolation as functional equations is investigated. These iterative relations are found in partial solution of the problem: To find all equations in iterative compositions of order $w = 1, 2, 3, \ldots$, satisfied by the elements of an abstract group, finite or infinite.

20. Dr. Ladd-Franklin maintains that one reason why the great work of Bertrand Russell—the reduction of all mathematics to a few pure-logic "primitives" (the word here proposed to cover at once indefinable terms and undemonstrable propositions)—has not been more widely accepted than it has is because his views are, hitherto, somewhat subject to change. His first book, the Foundations of Geometry, has long since been substantially discarded. Few who read the Principles of Mathematics (1903) realize the extent to which it has already been superseded by the Principia Mathematica (1910). In this interval it has been found necessary to give up not only the whole doctrine of classes but also that of propositional functions and even of relations in general. Insurmountable difficulties which presented themselves have been solved, at present, by the drastic "no classes theory," though it was thought at first (see *Proceedings of the London Mathematical Society*, 1905, for the very illuminating discussion) that the zigzaggedness theory or the small classes theory might serve the purpose better. One naturally waits a bit longer to see if the "no classes theory" may not also prove to be fallible.

A defect that certainly still remains in the doctrine of Peano and Russell—an example of the several infelicities of their form of symbol logic—is the so-called new relation "epsilon"; there is nothing peculiar in the *relation* concerned—the specificity lies simply in the subject term, which is "individual" or singular. The only reason they give for regarding the relation as peculiar—that it is not subject (as is the relation of

implication in general) to the rule of syllogism—is wholly fallacious; the real source of the difference lies in the change of sense of the middle term; that to confound in a middle term the sensus compositi and the sensus divisi is a source of danger has been a commonplace of logic since the time of the scholastics. That an inept symbolism is made use of in mathematics, which has for a fundamental interest the point and the "variable" (i. e., individuals, or singulars) would be of no consequence, but Russell and Peano treat this "addition" as constituting an important improvement over the logic which preceded them—that of Peirce and his school—instead of which it is simply erroneous.

F. N. COLE,
Secretary.

ON THE HEINE-BOREL PROPERTY IN THE THEORY OF ABSTRACT SETS.

BY DR. E. W. CHITTENDEN.

(Read before the American Mathematical Society, October 26, 1918.)

O. VEBLEN and N. J. Lennes have shown that in the presence of certain linear order axioms the Heine-Borel property is equivalent to the Dedekind cut axiom.* O. Veblen† and R. L. Moore‡ have used this property in systems of axioms for geometry and analysis situs.

M. Fréchet established the theorem that in a class (V) normale a subclass Ω has the Heine-Borel property if and only if Ω is compact and closed.§ This result was extended to systems $(\Omega; K^{1267})$ by T. H. Hildebrandt.||

E. R. Hedrick calls attention to the fact that the Heine-Borel theorem, in the enumerable case, is a consequence of the closure of derived classes.¶

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* Cf. O. Veblen, "The Heine-Borel theorem," this BULLETIN, vol. 10 (1904), p. 436–439.

† "A system of axioms for geometry," *Transactions Amer. Math. Society*, vol. 5 (1904), pp. 343–384.

‡ "On the foundations of plane analysis situs," ibid., vol. 17 (1916), p. 131.

§ Sur quelques points du calcul fonctionnel," *Rendiconti di Palermo*, vol. 22 (1906), p. 26.

|| "A contribution to the foundations of Fréchet's calcul fonctionnel," *Amer. Journal of Math.*, vol. 34 (1912), pp. 281–282.

¶ "On properties of a domain for which the derived set of any set is closed," *Transactions Amer. Math. Society*, vol. 12 (1911), pp. 285–294.

In a previous paper* I have shown that the Heine-Borel property implies self-compactness and extremality in any set $\mathfrak{L}$ of Fréchet. The present paper considers the Heine-Borel property in relation to the closure of derived classes and the property of separability. The converse of the theorem of Hedrick is established.

1. Let $\mathfrak{R}$ denote a fundamental set of elements called points such that for each point P and subset $\mathfrak{P}$ it is determined whether or not P is a limiting point of $\mathfrak{P}$. F. Riesz† regards the following four axioms as fundamental in the theory of systems $\mathfrak{R}$:

(*A*) If a point P is a limiting point of a set $\mathfrak{P}$ and $\mathfrak{P}$ is a subset of $\mathfrak{Q}$, then P is a limiting point of $\mathfrak{Q}$.

(*B*) If $\mathfrak{P} = \mathfrak{P}_1 + \mathfrak{P}_2$ and P is a limiting point of $\mathfrak{P}$, then P is a limiting point of $\mathfrak{P}_1$ or $\mathfrak{P}_2$.

(*C*) If P is a limiting point of $\mathfrak{P}$, then $\mathfrak{P}$ contains at least two elements.‡

(*D*) A point P is determined uniquely by the sets of which it is a limiting point.

A point P is *interior* to a set $\mathfrak{H}$ if $\mathfrak{H}$ contains P and a point of every set which has P for a limiting point.§

A family $(\mathfrak{H})$ of sets $\mathfrak{H}$ of points *covers* a set $\mathfrak{Q}$ if every point of $\mathfrak{Q}$ is *interior* to some class $\mathfrak{H}$.

A class $\mathfrak{Q}$ has the Heine-Borel property if every family $(\mathfrak{H})$ which *covers* $\mathfrak{Q}$ contains a finite subfamily

$$\mathfrak{H}_1, \mathfrak{H}_2, \mathfrak{H}_3, \cdots, \mathfrak{H}_n,$$

which also covers $\mathfrak{Q}$. The enumerable case of this property is defined by the restriction "if every *enumerable* family $(\mathfrak{H})$, etc." When the families $(\mathfrak{H})$ are not subject to this limitation I shall speak of the general case.

THEOREM 1. *If $\mathfrak{R}$ satisfies axiom (A) and $\mathfrak{P}$ is a subset of $\mathfrak{R}$ possessing the Heine-Borel property, then any closed subset of $\mathfrak{P}$ has the same property.*

* "On the converse of the Heine-Borel theorem in a Riesz domain," this BULLETIN, vol. 21 (1915), pp. 179–183. Cf. Theorems I and IV.

† Cf. "Stetigkeitsbegriff und abstrakte Mengenlehre," Atti del IV Congresso Inter. dei Mat., Roma (1908), vol. 2, pp. 18–24.

‡ From axioms (*B*) (*C*) it follows that, if $\mathfrak{P}$ has a limiting point, $\mathfrak{P}$ is an infinite class.

§ In the previous paper I assumed that this point be distinct from P. R. L. Root called my attention to the fact that this condition is unnecessary in view of axioms (*B*) and (*C*). The removal of the restriction permits a generalization of Theorem I of the previous paper, also suggested by Root.

Let $\mathfrak{Q}$ be a closed subset of $\mathfrak{P}$ and $(\mathfrak{H})$ be a family of sets covering $\mathfrak{Q}$. Denote by $\mathfrak{S}$ the class of all points of $\mathfrak{R}$ not contained in $\mathfrak{Q}$. From the closure of $\mathfrak{Q}$ and axiom (A) it follows that no subset of $\mathfrak{Q}$ has a limiting point in $\mathfrak{S}$. The family of sets obtained by adding $\mathfrak{S}$ to $(\mathfrak{H})$ covers $\mathfrak{P}$, and must contain a finite subfamily with the same property. This finite subfamily contains $\mathfrak{S}$, but $\mathfrak{S}$ contains no point of $\mathfrak{Q}$. Hence $(\mathfrak{H})$ contains a finite subfamily which covers $\mathfrak{Q}$, as was to be proved.

The following theorem is a generalization of theorem 1 of the previous paper, suggested to me by R. E. Root. As the proof of the present theorem is only slightly different from the proof of the original theorem, it is omitted here.

THEOREM 2. *If $\mathfrak{R}$ satisfies axiom (A) and $\mathfrak{P}$ is any subset of $\mathfrak{R}$ possessing the Heine-Borel property in the enumerable (or general case), then $\mathfrak{P}$ is self-compact; that is, every infinite subclass of $\mathfrak{P}$ has a limiting point in $\mathfrak{P}$.*

Theorems 1 and 2 are easily seen to hold for the general sets $(\mathfrak{B})$ recently introduced by Fréchet.*

2. It follows from theorem 2 that if $\mathfrak{R}$ satisfies axiom (A) and contains an infinite class with the Heine-Borel property then there is an element in $\mathfrak{R}$ which is a limiting point of an enumerable set. The following example shows that not all points of $\mathfrak{R}$ need have this property. Let $\mathfrak{R}$ denote the closed interval (0, 1) with the additional point $P = 2$. Limiting point is defined as usual for the interval (0, 1). The point $P = 2$, however, is to be a limiting point of every subset of $\mathfrak{R}$ which has the power of the continuum. Then $\mathfrak{R}$ satisfies axioms (A), (B), (C), (D), and also has the Heine-Borel property. Theorem 2 is valid, but the point $P = 2$ is not a limiting point of any enumerable set. The derived class of the rational points of $\mathfrak{R}$ is the interval (0, 1), which is not closed in $\mathfrak{R}$. Hence closure of derived classes is not implied by the Heine-Borel property in the set $\mathfrak{R}$ as conditioned by Riesz. If we assume in place of axiom (D) the stronger axiom:

(D') If P is a limiting point of a set $\mathfrak{P}$, then $\mathfrak{P}$ contains an enumerable set $\mathfrak{S}$, of which P is the only limiting point; we shall obtain a system $\mathfrak{R}$ in which closure of derived classes is implied by the Heine-Borel property. We have

THEOREM 3. *If $\mathfrak{R}$ satisfies axioms (A), (B), (C), (D') and any class $\mathfrak{Q}$ in $\mathfrak{R}$ possesses the Heine-Borel property in the enumerable case, the derived class of every subclass of $\mathfrak{Q}$ is closed.*

* *Comptes Rendus*, vol. 165 (1917), pp. 359–60.

We will first prove that $\mathfrak{Q}$ is closed. From axiom (D') any limiting point Q of $\mathfrak{R}$ is a unique limiting point of some enumerable subset $\mathfrak{S}$ of $\mathfrak{Q}$. $\mathfrak{S}$ is an infinite set (axioms (A), (B), (C)) and from theorem 2 has a limiting point in $\mathfrak{S}$. We shall now show that a contradiction results if we assume that $\mathfrak{Q}$ contains a set whose derived set is not closed. Let $\mathfrak{D}$ be a subset of $\mathfrak{Q}$ such that $\mathfrak{D}'$ has a limiting point Q not in $\mathfrak{D}'$. Then there is a sequence of points of $\mathfrak{D}'$,

$$\mathfrak{D}_1, \mathfrak{D}_2, \mathfrak{D}_3, \cdots$$

with the unique limiting point Q. Denote by $\mathfrak{H}_n$ the class of all points of $\mathfrak{R}$ excepting the points $\mathfrak{D}_{n+h}$ $(h = 1, 2, 3, \cdots)$, and $\mathfrak{Q}$. It is evident that each point of $\mathfrak{Q}$ is interior to at least one of the classes $\mathfrak{H}_n$. Denote by $\mathfrak{S}$ the set of all points not contained in the set $\mathfrak{D}$. Then the point Q is interior to S. The family

$$\mathfrak{S}, \mathfrak{H}_1, \mathfrak{H}_2, \cdots$$

is enumerable and covers the set $\mathfrak{Q}$. It contains a finite subfamily which also covers $\mathfrak{Q}$ and of which $\mathfrak{S}$ is a member. No class $\mathfrak{H}_n$ contains more than n of the points O_n. Therefore some point O_{n_0} must be interior to $\mathfrak{S}$. Since the point O_{n_0} is an element of the set $\mathfrak{D}'$, some point of $\mathfrak{D}$ must belong to $\mathfrak{S}$. This is the desired contradiction.

In a class $\mathfrak{L}$ of Fréchet* a point P is a limiting point of a class $\mathfrak{P}$ if and only if $\mathfrak{P}$ contains a sequence of distinct elements with the limit P. The class $\mathfrak{L}$ is a class $\mathfrak{R}$ satisfying axioms (A), (B), (C), (D'). From theorem 3 and a theorem of E. R. Hedrick† we have at once:

THEOREM 4. *In any set $\mathfrak{L}$ of Fréchet a necessary and sufficient condition that a subset $\mathfrak{P}$ of $\mathfrak{L}$ possess the Heine-Borel property is that $\mathfrak{P}$ be compact, closed, and that the derived set of every subset of $\mathfrak{P}$ be closed.*

3. F. Hausdorff‡ employs a form of the Heine-Borel property which does not imply the closure of derived sets in every system $\mathfrak{L}$ of Fréchet. The Hausdorff form of this property may be defined as follows: If a class $\mathfrak{P}$ is covered by a family $(\mathfrak{G})$ of regions§ $(\mathfrak{G})$ then $\mathfrak{P}$ is covered by a finite subset of $\mathfrak{G}$.

* *Rendiconti di Palermo*, loc. cit., p. 1, 2.

† Loc. cit., p. 286.

‡ Grundzüge der Mengenlehre, Veit and Co., Leipzig (1914), p. 231.

§ A region (Gebiet) is a set whose every element is an interior element. Cf. Hausdorff, loc. cit., p. 215, footnote 1.

Let the class $\mathfrak{L}$ be the interval (0, 1). We define limit as usual in (0, 1) with the single exception of the sequence of points $x = 1/m$ $(m = 2, 3, \ldots)$. The sequence $\Sigma(1/m)$ is to have the unique limit $x = 1$. Any subset $\mathfrak{P}$ of $\mathfrak{L}$ is compact, but derived classes are not closed. Let $\mathfrak{O}$ denote the irrational points of the interval (0, 1/2). Then $\mathfrak{O}'$ is the interval (0, 1/2), while $\mathfrak{O}''$ contains $x = 1$. Nevertheless it is easy to see that any closed subset of $\mathfrak{L}$ has the Hausdorff form of the Heine-Borel property. We give an example of a family $(\mathfrak{H})$ which covers $\mathfrak{L}$ and cannot be replaced by a finite subfamily.

Let $\mathfrak{H}_n$ be the class of all points of (0, 1) except the points $x = 1$ and $x = 1/m$ $(m > n)$. Denote by $\mathfrak{H}_0$ the class of all points not an irrational point of the interval (0, 1/2). Then

$$\mathfrak{H}_0, \mathfrak{H}_1, \ldots, \mathfrak{H}_n, \ldots$$

is an enumerable family which covers $\mathfrak{P}$. No class $\mathfrak{H}_n$ contains more than a finite number of the points $x = 1/m$, and no point $x = 1/m$ $(m > 1)$ is interior to $\mathfrak{H}_0$. It follows that the family $(\mathfrak{H})$ cannot be replaced by a finite subfamily. The set $\mathfrak{H}_0$ is not a region.

4. Fréchet* has studied classes $(\mathfrak{V})$ in which for each element there is given a family $(\mathfrak{V})$ of neighborhoods (voisinages). We hall impose the following conditions on the families $(\mathfrak{V})$: (1) For every element (point) P the corresponding family $(\mathfrak{V})$ is enumerable and is denoted by $(\mathfrak{V}^m)$; (2) the point P is contained in every neighborhood $\mathfrak{V}^m$ of P and is the only such point: (3) in a family $(\mathfrak{V}^m)$, $\mathfrak{V}^m$ contains $\mathfrak{V}^{m+1}$; (4) if a point Q is in a neighborhood $\mathfrak{V}^m$ of P then P is in the corresponding neighborhood $\mathfrak{V}^m$ of Q.†

A point P is a limiting point of a class $\mathfrak{P}$ if every neighborhood of P contains a point of $\mathfrak{P}$ distinct from P.

THEOREM 5. *If $\mathfrak{Q}$ is a set in a system $(\mathfrak{V})$ with the Heine-Borel property, and if $\mathfrak{Q}'$ contains $\mathfrak{Q}$, then $\mathfrak{Q}$ is compact, perfect, and separable.*

Since $\mathfrak{Q}'$ contains $\mathfrak{Q}$ and $\mathfrak{Q}$ is compact (theorem 2) it follows that $\mathfrak{Q}$ is perfect. We have to prove that $\mathfrak{Q}$ contains an enumerable subset $\mathfrak{E}$ such that $\mathfrak{Q} = \mathfrak{E}'$.

Each point P is interior to the corresponding neighborhood

* *Comptes Rendus*, loc. cit.

† The class $(\mathfrak{V})$ admits a definition of a symmetric distance function δ, such that $\delta(P, Q) = 0$ if and only if $P = Q$.

$\mathfrak{V}^m$. The system of neighborhoods $\mathfrak{V}^m$ for fixed m covers $\mathfrak{Q}$ and may be replaced by a finite subsystem,

$$\mathfrak{V}^{m1}, \quad \mathfrak{V}^{m2}, \quad \ldots \quad \mathfrak{V}^{mk};$$

such that each point of $\mathfrak{Q}$ is interior to some class $\mathfrak{V}^{mk}$ and each class $\mathfrak{V}^{mk}$ is a neighborhood of a point Q^{mk} of the class $\mathfrak{Q}$. Let $\mathfrak{E}$ be the class of all points Q^{mk}. Since every point P of $\mathfrak{Q}$ is interior to some set $\mathfrak{V}^{mk}$, it follows from condition (4) that Q^{mk} is contained in the neighborhood $\mathfrak{V}^m$ of P. Therefore P is a limiting point of the class $\mathfrak{E}$.

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This article was in type before the writer learned of the existence of an article by Fréchet (*Bulletin de la Societé mathématique de France*, volume 35, 1917), in which it is shown that the *closure* of *derived classes* is a consequence of the Heine-Borel property in the case of a general system $\mathfrak{L}$. Theorem 3 of the present paper is a generalization of this result.

E. W. CHITTENDEN.

INTEGRALS AROUND GENERAL BOUNDARIES.

BY PROFESSOR P. J. DANIELL.

THE concept of a boundary integral has been extended to curves of the type $x = x(t)$, $y = y(t)$, where $x(t)$, $y(t)$ are absolutely continuous functions of a parameter t. In this case the curves are more or less simple and have tangents "nearly everywhere." In applications to physics however the boundary must be considered rather as a boundary of a set (in the sense of the theory of point sets). The boundary will be, in general, a collection of points without definite tangents at all. This paper sets out a method by which such boundary integrals can be defined under certain restrictions placed on the two integrand functions u, v. The method depends on the concept of absolutely additive functions of sets.* The writer believes that these restrictions could be lightened and that there is a wide field here for further investigation.

* J. Radon, *Wiener Sitzungsberichte*, vol. 122 (1913), p. 1295. W. H. Young, *Proceedings London Math. Society*, vol. 13 (1914), p. 109.

Statement of Problem.—Given any set E measurable Borel, and its boundary $B(E)$, contained in a closed fundamental interval J $(0 \leqq x \leqq 1, 0 \leqq y \leqq 1)$; given also two functions $u(x, y)$, $v(x, y)$ summable in the interval J; to define

$$\int_{B(E)} u\,dx + v\,dy.$$

Note.—The boundary integral is taken in such a sense that on a rectangle for the side with the lesser value of y the integral is taken in the direction of x increasing. As the axes are usually drawn this corresponds to a counter-clockwise sense.

Definitions and Restrictions.—

R 1. Let the total variation of $u(x, y)$, varying y, be $\lambda(x)$, where $\lambda(x)$ is finite nearly everywhere in x and summable in $(0 \leqq x \leqq 1)$.

R 2. Let the total variation of $u(x, y)$, varying x, be $\mu(y)$, where $\mu(y)$ is finite nearly everywhere in y and summable in $(0 \leqq y \leqq 1)$.

We shall consider in the first place rectangles r with sides parallel to the axes. Then

$$\int_{B(r)} udx + vdy = \int_r d\alpha(x, y)$$

can be proved to be an absolutely additive function of rectangles r, and we may define

$$\int_{B(E)} udx + vdy = \int_E d\alpha(x, y).$$

If $d\alpha(x, y)$ is an absolutely additive function of rectangles we can by Radon's method define $\int_E d\alpha(x, y)$ uniquely for any set E measurable Borel contained in J. All that is needed then is to prove that $\int_{B(r)} udx + vdy$ is an absolutely additive function of rectangles r and to define

$$\alpha(x, y) = \int_{B(r')} ud\xi + vd\eta,$$

where r' is the rectangle $(0 < \xi < x, 0 < \eta < y)$.

Proof. By R_1, the total variation of $u(x, y)$ in $(0 \leqq y \leqq 1)$

is $\lambda(x)$. Denote the total variation in $(0 \leqq \eta \leqq y)$ by $\lambda(x)$ $\theta(x, y)$ when $\lambda(x)$ is finite. $\lambda(x)$ is non-negative, $\theta(x, y)$ is non-negative and a non-decreasing function of y taking the value 0 when $y = 0$, and 1 when $y = 1$. In particular it is a limited measurable function of x. Define

$$\sigma_1(x, y) = \int_0^x \lambda(x)\theta(x, y)dx.$$

$$\int_r d\sigma_1(x, y) = \sigma_1(x_1, y_1) + \sigma_1(x_2, y_2) - \sigma_1(x_1, y_2) - \sigma_1(x_2, y_1)$$

$$= \int_{x_1}^{x_2} \lambda(x)dx[\theta(x, y_2) - \theta(x, y_1)]$$

is a finite non-negative additive function of rectangles. Then for any set E measurable Borel

$$\int_E d\sigma_1(x, y) \text{ is defined and } \leqq \int_J d\sigma_1(x, y) \text{ or } S_1.$$

By R_1,

$$|u(x, y_1) - u(x, y_2)| \leqq \lambda(x)[\theta(x, y_2) - \theta(x, y_1)].$$

$u(x, y)$ is summable in (x, y) or is summable in x for nearly all values of y. Let y_0 be one of the values for which it is summable. Then

$$|u(x, y)| \leqq |u(x, y_0)| + \lambda(x)|[\theta(x, y) - \theta(x, y_0)]|$$

or $u(x, y)$ is summable in x for *all* values of y.

$$\left|\int_{B(r)} udx\right| = \left|\int_{x_1}^{x_2} [u(x, y_1) - u(x, y_2)]dx\right|$$

$$\leqq \int_{x_1}^{x_2} \lambda(x)[\theta(x, y_2) - \theta(x, y_1)]$$

or

$$\int_r d\sigma_1(x, y).$$

Hence for any set $\sum_{i=1}^{n} r_i$ of non-overlapping rectangles

$$\sum_{i=1}^{n} \left|\int_{B(r_i)} udx\right| \leqq \int_{\Sigma_1^n r_i} d\sigma_1(x, y)$$

$$\leqq S_1.$$

Therefore

$$\sum_{i=1}^{\infty} \int_{B(r_i)} u dx$$

is absolutely convergent.

Moreover $\int_{B(r)} u dx$ is an additive function of rectangles r, for if two or more rectangles have some parts of their boundaries in common (but do not overlap), the integrals along these parts being taken in opposite directions will annul each other.

If we define

$$\alpha_1(x, y) = \int_{B(r')} u dx, \quad r' = (0 \text{ to } x, 0 \text{ to } y),$$

$$\int_{B(r)} u dx = \int_r d\alpha_1(x, y)$$

defines an absolutely additive function of rectangles. Similarly for

$$\int_{B(r)} v dy = \int_r d\alpha_2(x, y)$$

and therefore also for

$$\int_{B(r)} u dx + v dy = \int_r d\alpha(x, y),$$

where $\alpha(x, y) = \alpha_1(x, y) + \alpha_2(x, y)$.

This was to be proved, and it follows that we can define

$$\int_{B(E)} u dx + v dy = \int_E d\alpha(x, y).$$

More generally, the same method could be used if it can be proved that $\int_{B(r)} u dx + v dy$ is an absolutely additive function of rectangles; the difficulty is to state the required conditions as conditions on u and v directly. That is the reason for the introduction of R_1, R_2, which are sufficient but probably not necessary.

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DETERMINANT GROUPS.

BY PROFESSOR G. A. MILLER.

(Read before the American Mathematical Society April 13, 1918.)

§1. *Introduction.*

LET D represent a determinant of order n whose n^2 elements are regarded as independent variables. The substitutions on these n^2 elements which transform D into itself constitute a substitution group G, which we shall call the *determinant group* of degree n^2. As the elements of D are supposed to be independent variables, it results from the definition of a determinant that every substitution of G must transform the elements of D in such a manner that all the elements of a line (row or column) appear in a line after the transformation.

Hence the substitutions of G correspond to the permutations of the elements of D resulting from transforming its rows and columns independently according to the alternating group of degree n, transforming its rows and columns simultaneously according to negative substitutions in the symmetric group of this degree, and interchanging the rows and columns. The order of G is therefore $(n!)^2$, and hence the number of the distinct determinants that can be formed by permuting the n^2 elements of D is $n^2!/(n!)^2$.* These determinants may be arranged in pairs such that each pair is composed of the determinants which differ only with respect to sign. In particular, the square of D is transformed into itself by a group K whose order is twice the order of G and which contains G as an invariant subgroup.

Some of the abstract properties of G follow directly from the fact that it is simply isomorphic with the imprimitive substitution group of degree $2n$ whose head is composed of the positive substitutions in the direct product of two symmetric groups of degree n. These substitutions correspond to interchanges of the rows among themselves and the columns among themselves or a combination of such interchanges. The re-

* G. Bagnera, *Giornale di Matematiche*, vol. 25 (1887), p. 228; it may be noted that in the review of this article in *Jahrbuch über die Fortschritte der Mathematik* the author's name appears in the form Bergnera.

maining substitutions of this imprimitive group correspond to the interchanges of rows and columns. Hence it results directly that *G contains exactly three subgroups of index 2 for every value of n.*

When $n > 4$, G contains no invariant subgroup besides the three of index 2 mentioned above and their common subgroup of index 4 under G, which is also the commutator subgroup of G. In this case G is unsolvable and its factors of composition are $2, 2, n!/2, n!/2$. When $n = 2$, G is simply isomorphic with the four-group, when $n = 3$ it is simply isomorphic with the square of the non-cyclic group of order 6, and when $n = 4$ it contains an invariant abelian subgroup of order 16 and of type (1, 1, 1, 1) which is complementary to a quotient group of order 36 simply isomorphic with the square of the non-cyclic group of order 6.

§ 2. *Determinant Group as a Substitution Group of Degree n^2.*

It has been noted that the substitution group of degree $2n$ which is simply isomorphic with G and corresponds to the permutation of the rows and columns of D is always imprimitive. On the other hand, it is easy to prove that G itself is intransitive when $n = 2$, imprimitive when $n = 3$, and both simply transitive and primitive for every value of n which exceeds 3. In fact, the cases when $n = 2$ or 3 can readily be established by means of the well-known lists of possible substitution groups of degrees 4 and 9.*

That G is always simply transitive when $n > 3$ results directly from the fact that all its substitutions which omit a given letter constitute a subgroup having two transitive constituents. One of these is on the $2n - 2$ other letters in the row and column which contain the fixed letter, while the other transitive constituent is on the $(n - 1)^2$ letters not found in this row or this column. The fact that G is primitive is a consequence of the theorem that *if the subgroup composed of all the substitutions which omit one letter of an imprimitive group omits only one letter, then the degree of one of the transitive constituents of this subgroup increased by one must divide the degree of the imprimitive group.*

The group G contains exactly two substitutions which trans-

* F. N. Cole, *Bulletin of the New York Mathematical Society*, vol. 2 (1893), p. 252.

form each of the elements of any term of D into itself. In particular, the principal term of D is transformed into itself by the substitution of order 2 which interchanges the kth row of D and the kth column, $k = 1, 2, \ldots, n$. Each of the terms of D is therefore transformed into itself by at most $2 \cdot n!$ of the substitutions of G. As such a term cannot be transformed under G into more than $n!/2$ terms, it results that this is the actual number of the substitutions of G which transform a term into itself. The elements of each term of D must therefore be transformed under G according to the symmetric group of degree n.

The positive terms of D, as well as the negative ones, are transformed under G according to a simply transitive group of degree $n!/2$ since the order of G is not divisible by $n!/2 - 1$, whenever $n > 3$. The two substitutions of G which transform each of the elements of one of these terms into itself constitute a group of degree $n(n-1)$, hence G involves two complete sets of $n!/2$ conjugate substitutions of order 2 and of degree $n(n-1)$. These sets form a single set of conjugates under the group of D^2 and each of their substitutions is invariant under a group which is simply isomorphic with the direct product of the symmetric group of degree n and a group of order 2. As this group of order $2 \cdot n!$ is a maximal subgroup of G, it results that the simply transitive groups of degree $n!/2$ according to which the positive and the negative terms of D are transformed under G must be primitive whenever $n > 3$.

It was noted above that the subgroup composed of all the substitutions of G which omit a given letter has two transitive constituents of degrees $2(n-1)$ and $(n-1)^2$ respectively. This subgroup is evidently formed by a simple isomorphism between these constituent groups and hence each of these groups has for its order the square of $(n-1)!$. The former constituent is imprimitive, being formed by extending, by a substitution of order 2 which merely interchanges the corresponding letters, the positive substitutions in the direct product of two symmetric groups of degree $n-1$ written on distinct sets of letters. The latter constituent is the group of the determinant of order $n-1$ and hence is a simply transitive primitive group, whenever $n > 4$. In particular, *the subgroup of the determinant group of degree n^2 which is composed of all its substitutions omitting a given letter is simply isomorphic with the group of the determinant of order $n-1$ and has this group for a transitive constituent whenever $n > 3$.*

Although G is a simply transitive primitive group whenever $n > 3$, it always contains a subgroup of index 2 which is imprimitive, viz., the subgroup corresponding to the interchanges of the rows and columns according to the positive substitutions in the square of the symmetric group. In fact, this subgroup has two sets of systems of imprimitivity, one composed of the elements of rows and the other composed of elements of columns. Each system of the first set has one and only one element in common with each system of the second set. This subgroup contains no other set of systems of imprimitivity and these two sets are transformed into each other under the primitive group G.

The simply transitive primitive group G is evidently of class $3n$, $n > 3$, and hence we have here an interesting infinite system of simply transitive primitive groups in which there are groups for which the ratio of the degree to the class exceeds any given finite number, this ratio being $n/3$. In § 4 we shall consider another infinite system of such groups for which this ratio is still larger. When a primitive group is at least doubly transitive it is well known that this ratio cannot exceed 4 unless the group is either alternating or symmetric.*

§ 3. *Determinant Groups as Substitution Groups of Degree* $n!/2$.

When $n = 2$, D has only one positive term, which is therefore invariant under G. When $n = 3$, D has three positive terms, which are transformed under G according to the symmetric group of degree 3 and therefore each of these three terms is transformed into itself by six of the substitutions of G. These six substitutions transform the three negative terms of D according to the symmetric group of degree 3, hence G contains no substitution besides identity which is commutative with each of the six terms of D.

It has been noted that G transforms the $n!/2$ positive terms of D according to a simply transitive primitive group G', which is simply isomorphic with G whenever $n > 3$ because G involves no invariant subgroup whose index exceeds 4. Let T_1 be the principal diagonal term of D, and let T_2 be a positive term of D which has the last $n - 3$ elements in common with T_1 but is not identical with T_1. Among the $2 \cdot n!$ substitutions of G which transform T_1 into itself there are

* Cf. Encyclopédie des Sciences mathématiques, Tribune publique 14, no. 299.

$6\cdot(n-3)!$ which also transform T_2 into itself. The subgroup of G composed of all its substitutions which omit a given letter must therefore contain a transitive constituent of degree $[n(n-1)(n-2)]/3$.

The number of the transitive constituents of this subgroup varies with n and increases without limit as n increases without limit, since the positive terms which have α elements in common with T_1 can clearly not be transformed under this subgroup into those which have $\beta \neq \alpha$ elements in common with T_1. When $n > 4$ each of these constituents is either of order $n!$ or of twice this order, because this subgroup has only one invariant subgroup whose index exceeds 4 and could not have a transitive constituents of order 2 or of order 4. In fact, each term which has an element in common with T_1 has evidently more than four conjugates under this subgroup whenever $n > 4$, and if a term has no element in common with T_1 its element which occurs in the first row, for instance, can be transformed into $n-1$ other elements of this row under this subgroup.

Among the substitutions which transform T_1 into itself there are $2^{e+1}\cdot e!$ substitutions which also transform the secondary diagonal term of D into itself, e being the largest integer which does not exceed $n/2$. Hence it results that the secondary diagonal term of D is always transformed into an odd number of conjugates under the group formed by the substitutions which transform T_1 into itself, $2\cdot n! \div 2^{e+1}\cdot e!$ being an odd integer as can readily be proved. In particular, when the secondary diagonal term of D is positive the subgroup of G' composed of all its substitutions which omit a given letter must contain a transitive constituent of odd degree and hence of order $n!$ whenever $n > 4$. It may be noted in passing that the given considerations furnish also a proof of the known theorem that the continued product $(m+1)(m+2)\ldots(2m)$ is always divisible by 2^m.

§ 4. *Group of the Square of a Determinant.*

In § 1 it was noted that the substitution group K on the n^2 elements of D composed of all the substitutions on these elements which transform D^2 into itself is of order $2(n!)^2$ and contains G invariantly. Its factors of composition are therefore 2, 2, 2, $n!/2$, $n!/2$. When $n = 2$, it is the octic group and

hence it is imprimitive. For all other values of n it is a simply transitive primitive group. In fact, it could not be doubly transitive, since the subgroup composed of all its substitutions which omit a given element has evidently two transitive constituents of degrees $2n - 2$ and $(n-1)^2$ respectively. Moreover, it could not be imprimitive when n exceeds 3, since it contains the primitive group G which is also of degree n^2. In the special case when $n = 3$, K is clearly again primitive since its subgroup composed of all its substitutions omitting one letter has two transitive constituents of degree 4, and 9 is not divisible by $4 + 1$.

One of the most interesting facts connected with the infinite system of simply transitive primitive groups represented by K is that these groups are of a very low class. It was noted near the close of § 2 that the class of a primitive group which is at least doubly transitive and does not include the alternating group cannot be less than its degree divided by 4; but when a primitive group is only simply transitive C. Jordan already observed that the ratio of the degree to the class has no upper limit. For the system of simply transitive primitive groups used by him as an illustrative example this ratio becomes however only about one half as large as in the present case since *K is a simply transitive primitive group of degree n^2 and of class $2n$ whenever $n > 2$.**

As a transitive substitution group on the rows and columns of D, K is clearly the largest possible imprimitive group of degree $2n$ which has two systems of imprimitivity and hence it includes all the possible imprimitive groups of degree $2n$ which contain two such systems. As a transitive group on the $n!$ terms of D it is also imprimitive when $n > 2$, since it transforms the positive and negative terms of D into each other and involves a subgroup of index 2 which transforms these terms respectively among themselves. When $n > 3$ this subgroup is obtained by establishing a simple isomorphism between two simply transitive primitive groups, when $n = 3$ it is the direct product of two symmetric groups of degree n, and when $n = 2$ it is identity.

Since K contains a subgroup of index 2 which is simply isomorphic with the square of the symmetric group of degree n, it results directly that its smallest invariant subgroup is of

* Cf. W. A. Manning, *American Journal of Mathematics*, vol. 32 (1910), p. 256.

index 8 whenever $n > 4$, and that the corresponding quotient group is the octic group. Hence K involves exactly three subgroups of index 2 whenever n exceeds 4 and only two other invariants subgroups besides identity, viz., the mentioned subgroup of index 8 and one of index 4 corresponding to the invariant subgroup of order 2 of the octic group. These results apply also to the special case when $n = 3$.

TRANSLATION SURFACES IN HYPERSPACE.

BY PROFESSOR C. L. E. MOORE.

(Read before the American Mathematical Society, April 27, 1918.)

1. If the rectangular coordinates of the points of a surface can be expressed in the parametric form

$$x_i = f_i(u) + g_i(v) \qquad (i = 1, 2, \ldots, n), \tag{1}$$

where f_i are functions of u alone and g_i functions of v alone, the surface is called a translation surface. It is seen that a translation can be found which will send any parameter curve $u = \text{const.}$ into any other one of the same system. The same is true of the curves $v = \text{const.}$ The surface (1) is also seen to be the locus of the mid-points of the lines joining the points of

$$C_1\colon\ x_i = 2g_i(u) \quad \text{to the points of} \quad C_2\colon\ x_i = 2f_i(v). \tag{2}$$

The character of the surface can then be determined, in a great measure, by the form and relative position of these two curves. Nearly all writers on surface theory* mention three facts concerning translation surfaces in 3-space:

(*a*) The generators of the developable which touches the surface along a curve $u = \text{const.}$ are tangent to the curves $v = \text{const.}$, or in other words the directions of the parameter curves passing through a given point are conjugate directions.

(*b*) There are surfaces which can be expressed in more than one way in the form (1).

* Darboux, Théorie générale des Surfaces, vol. 1, pp. 148, 340. Scheffers, Theorie der Flächen, vol. 2, pp. 188, 245.

(*c*) Minimum surfaces are always translation surfaces. The curves $u = \text{const.}$, $v = \text{const.}$, in this case, are minimum curves.

It is the object of the present note to examine translation surfaces in hyperspace as to the properties (*a*), (*b*), (*c*) and also to determine the relation between the curves C_1 and C_2 in order that (1) be a developable surface.

From (1) we see that the coordinates of the surface satisfy the linear partial differential equation of the second order

$$\frac{\partial^2 Z}{\partial u \partial v} = 0. \tag{3}$$

Segre* showed that if the coordinates of a surface satisfy a linear partial differential equation of the second order, then there are two directions through each point having the property (*a*) of conjugate directions. These are called the *characteristics* and are determined by the characteristic equation of the partial differential equation. The characteristic equation of (3) is

$$du\,dv = 0.$$

Therefore the parameter curves on (1) are characteristics. Hence *if* (3) *is the only partial differential equation of the second order which the coordinates of the surface satisfy, there are just two characteristics through each point and the surface cannot be expressed in more than one way in the form* (1). The form of equation (1) shows that the developables mentioned in (*a*) are cylinders.

2. *Ruled Translation Surfaces.*—Segre showed, in the paper referred to, that if the coordinates of a surface satisfy two linear partial differential equations of the second order the surface must either lie in a 3-space or else consist of the tangent lines to a twisted curve. I showed† that if the surface does not lie in a 3-space the two differential equations must be such that the pencil formed by them will contain only one equation of parabolic type. If now in addition to (3) the coordinates

* "Su una classa di superficie degl'iperspazii," ecc. *Atti di Torino*, 1907.

† C. L. E. Moore, "Surfaces in hyperspace which have a tangent line with three-point contact passing through each joint." This BULLETIN, vol. 18 (1912).

satisfy a second equation, it must be of the form

$$A\frac{\partial^2 Z}{\partial u^2}+B\frac{\partial Z}{\partial u}+C\frac{\partial Z}{\partial v}=0. \tag{4}$$

Every equation of the pencil

$$A\frac{\partial^2 Z}{\partial u^2}+\lambda\frac{\partial^2 Z}{\partial u\partial v}+B\frac{\partial Z}{\partial u}+C\frac{\partial Z}{\partial v}=0,$$

where λ is the parameter, will then be satisfied. The surface will then have an infinite number of characteristics. It is to be observed, however, that the characteristic equation is

$$Adv^2-\lambda dudv=0,$$

which for all values of λ has $dv=0$ for one factor. This is the direction of the characteristic of (4), and I showed in the article referred to that this is the direction of three-point contact and that there is only one such direction passing through a point and hence this must be the direction of the rulings of the developable. We therefore conclude that however the surface is expressed in the form (1) the rulings must form one parameter system. The other parameter system can be any one-parameter family of curves traced on the developable. Equation (1) then takes the form

$$x_i=k_iu+g_i(v),$$

where k_i are constants. It is seen that the rulings are parallel and therefore the surface is a cylinder. Hence *cylinders are the only translation surfaces in hyperspace which can be expressed in the form* (1) *in more than one way.*

The coordinates of any ruled surface must satisfy a parabolic differential equation of the second order, and if it is a translation surface will satisfy two equations and therefore will be the same as above. Hence *cylinders are the only ruled surfaces of translation.*

3. *Developables.*—Let u and v be the arc length on the parameter curves. Then the element of arc on the surface becomes

$$ds^2=du^2+a_{12}dudv+dv^2,$$

where $a_{12} = \Sigma f_i' g_i'$. The formula for the Gaussian curvature then reduces to

$$\sqrt{a}G = \frac{\partial}{\partial v}\left[\frac{2}{\sqrt{a}}\frac{\partial a_{12}}{\partial u}\right], \tag{5}$$

where $a = 1 - a_{12}^2$. If $G = 0$, we have on integrating

$$a_{12} = \cos(U + V), \tag{6}$$

where U is an arbitrary function of u alone and V an arbitrary function of v alone. This is then the condition that the surface be developable, that is, have zero curvature. From the definition of a_{12} we see that (6) says that the angle between any tangent to C_1 and any tangent to C_2 is always expressible as a function of u plus a function of v. To determine the relation between C_1 and C_2 in order that (6) may be satisfied let γ_1 and γ_2 be the spherical representation of the curves C_1 and C_2. (That is, γ_1 and γ_2 are the traces on the unit hypersphere, center at O say, made by lines through O parallel to the tangents.) The distance from a point of γ_1 to a point of γ_2, measured on the sphere, will be equal to the angle between the corresponding tangents to C_1 and C_2. Let u_i represent a series of points on γ_1 and v_i a series of points on γ_2. Let θ_{ij} represent the distance from u_i to v_i. Then, if relation (6) is satisfied, we have

$$\theta_{ri} = U(u_r) - V(v_i), \quad \theta_{si} = U(u_s) - V(v_i).$$

Subtracting,

$$\theta_{ri} - \theta_{si} = U(u_r) - U(u_s).$$

Thus the difference of the distances from v_i to u_r and u_s is independent of v_i and hence v_i must lie in the locus of points the difference of whose distances from u_r and u_s is constant. This must be true for any values of r and s. Hence if γ_1 is an n-dimensional curve (does not lie in a space of lower dimensions), γ_2 must reduce to a point, which means that C_2 is a straight line. If however γ_1 lies in a linear space $S_k (k < n)$ passing through O, then (6) will be satisfied if γ_2 lies in the space S_{n-k} completely perpendicular to S_k. In this case the distance from any point of γ_1 to any point of γ_2 is $\pi/2$. C_1 and C_2 will then be any two curves lying in completely per-

pendicular spaces. In particular if γ_1 is a great circle on the hypersphere, then γ_2 could either lie in the completely perpendicular space or it could coincide with γ_1, in which case (6) would be satisfied. The curves C_1 and C_2 for this last condition become any curves lying in parallel planes. The surface of translation would then reduce to a plane. If now γ_1 lies in a linear space S_k' which does not pass through O and γ_2 lies in the space S_{n-k}' passing through O and completely perpendicular to S_k', then each point of γ_2 will be the same distance from all points of γ_1 but these distances will vary with the point of γ_2. In this case then the distance is a function of v only and (6) will be satisfied. The curve C_2 will then be any curve lying in a space of $n - k$ dimensions and C_1 will be such that any tangent will make the same angles with any line in this space. If the space of C_2 be taken as a coordinate space the two curves must then have the form

$$(7)\qquad \begin{aligned} C_1:\ & x_i = g_i(u) && (i = 1, 2, \ldots, k), \\ & x_j = a_j u && (j = k+1, k+2, \ldots, n); \\ C_2:\ & x_i = 0 && (i = 1, 2, \ldots, k), \\ & x_j = f_j(v) && (j = k+1, k+2, \ldots, n), \end{aligned}$$

where a_j are constants. We then have the results: (α) *if C_1 and C_2 lie in parallel planes, the resulting translation surface is a plane;* (β) *if they lie in completely perpendicular planes, or* (γ) *if they have the form* (7), *the translation surface is a non-ruled developable.* (δ) *If one of the curves reduces to a straight line, the translation surface is ruled.*

The surface generated by the midpoints of the lines joining the points of the two curves

$$x_1 = 2a\cos u, \quad x_2 = 2a\sin u, \quad x_3 = x_4 = 0.$$
$$x_1 = x_2 = 0, \quad x_3 = 2b\cos v. \quad x_4 = 2b\sin v$$

is the rotation surface

$$x_1 = a\cos u, \quad x_2 = a\sin u, \quad x_3 = b\cos v, \quad x_4 = b\sin v$$

which is left invariant by all the rotations leaving the x_1x_2 and x_3x_4 planes invariant.*

*C. L. E. Moore, "Rotations in hyperspace," *Proceedings American Academy*, vol. 53 (1918).

4. *Covariant Derivatives.*—In discussing those properties of a surface that have to do with the normals or curvature of curves traced on the surface it is very convenient to make use of covariant derivatives instead of the ordinary second partial derivatives. If the surface is expressed in vector form, the covariant derivatives are always normal to the surface while the ordinary second partial derivatives are not. A second vector fundamental form, analogous to the second fundamental form in three dimensions can be written in terms of these covariant derivatives.* The vector equation of (1) is

$$\rho = \sum_1^n (f_i(u) + g_i(v))k_i,$$

where k_i are unit vectors parallel to the coordinate axes. Then

$$(8) \qquad m = \frac{\partial \rho}{\partial u} = \Sigma f_i'(u)k_i, \quad n = \frac{\partial \rho}{\partial v} = \Sigma g_i'(v)k_i.$$

If u and v are arc lengths along the parameter curves,

$$(9) \quad m^2 = m \cdot m = \Sigma f_i'^2(u) = 1, \quad n^2 = n \cdot n = \Sigma g_i'^2(v) = 1.$$

The ordinary second partial derivatives are

$$(10) \qquad p = \frac{\partial^2 \rho}{\partial u^2} = \Sigma f_i''(u)k_i, \quad q = \frac{\partial^2 \rho}{\partial u \partial v} = 0, \quad r = \frac{\partial^2 \rho}{\partial v^2} = \Sigma g_i''(v)k_i.$$

From these we obtain

$$(11) \qquad \begin{aligned} m \cdot p &= \Sigma f_i' f_i'' = 0, \quad m \cdot q = 0, \quad m \cdot r = \Sigma f_i' g_i'', \\ n \cdot p &= \Sigma f_i'' g_i', \quad n \cdot q = 0, \quad n \cdot r = \Sigma g_i' g_i'' = 0. \end{aligned}$$

In terms of these we can write the covariant derivatives†

$$(12) \quad y_{11} = (m \times n) \times (m \times n \times p) = \frac{1}{a} \begin{vmatrix} m & n & p \\ m^2 & m \cdot n & m \cdot p \\ n \cdot m & n^2 & n \cdot p \end{vmatrix},$$

* Wilson and Moore, "Differential geometry of two dimensional surfaces in hyperspace," *Proceedings American Academy*, vol. 52 (1916).

† Wilson and Moore, loc. cit., pp. 337–338.

where $a = 1 - (\Sigma f'g')^2$. The derivative y_{12} is obtained by replacing p by q in (12) and y_{22} is obtained by replacing p by r. It is seen that $y_{12} = 0$. This is a sufficient condition that the plane of the indicatrix (the locus of the end of the curvature vector of normal sections of the surface at a given point) pass through the surface point. Translation surfaces therefore are what Wilson and Moore called of the four-dimensional type.

5. If now the curves C_1 and C_2 lie in completely perpendicular spaces,

$$m \cdot n = \Sigma f_i' g_i' = 0.$$

This is the only surface of translation on which the parameter curves are orthogonal. The vector p will lie in the plane of C_1, and r in the plane of C_2, therefore

$$m \cdot r = \Sigma f_i' g_i'' = 0, \qquad n \cdot p = \Sigma f_i'' g_i' = 0.$$

Hence we have for this type of developable

$$y_{11} = p, \qquad y_{12} = 0, \qquad y_{22} = r. \tag{13}$$

Thus p and r are normal to the surface and since they are the curvatures of the parameter curves we see that the curves $u = \text{const.}$, $v = \text{const.}$ are geodesics on the surface (curvature lies in the normal plane). Hence *on a non-ruled developable translation surface on which the generating curves are everywhere orthogonal those curves are geodesics.*

The area of the indicatrix is (Wilson and Moore, page 333)

$$2a^{3/2}\mu \times \delta = a_{12}y_{11} \times y_{22} + (a_{22}y_{11} - e_{11}y_{22}) \times y_{12};$$

and since $a_{12} = y_{12} = 0$, the area is zero. The indicatrix then reduces to a linear segment. Since y_{11} and y_{22} do not coincide in direction, this linear segment does not pass through the surface point. The curves $u = \text{const.}$, $v = \text{const.}$ are the Segre characteristics, and we know that when the indicatrix reduces to a linear segment the characteristics will be orthogonal and that the end of the curvature vector of these curves are the ends of the linear segment to which the indicatrix reduces. The indicatrix subtends a right angle at the surface point.

The vector mean curvature is half the sum of the vector

curvatures in two perpendicular directions passing through the point considered. This sum is independent of the pair of perpendicular directions chosen. Then in this case we can choose the parameter curves and we have

$$h = \tfrac{1}{2}\Sigma(f_i'' + g_i'')k_i,$$

from which we see that the locus of the end of the mean curvature vector is also a translation surface and in fact is a developable of the type of the original surface. The locus of the indicatrix as the point in question describes the whole surface is the locus of lines cutting the two curves

$$C_1'': \quad x_i = 2f''(u), \qquad C_2'': \quad x_i = 2g_i''(v).$$

If the mean curvature is perpendicular to the indicatrix,

$$\Sigma f_i''^2 - \Sigma g_i''^2 = 0,$$

hence the curvature of C_1 and C_2 must be the same at all points and both constant, that is, they must be equal circles. If they have the same center the surface will be a rotation surface.

6. The second kind of non-ruled developable arises when

$$\begin{aligned} C_1: \quad & x_i = 2f_i(u) && (i = 1, 2, \ldots, k), \\ & x_j = 2a_j u && (j = k+1, k+2, \ldots, n); \\ C_2: \quad & x_i = 0 && (i = 1, 2, \ldots, k), \\ & x_j = 2g_j(v) && (j = k+1, k+2, \ldots, n), \end{aligned}$$

and the equation of the surface will be

$$\rho = \sum_1^k f_i(u)k_i + \sum_{k+1}^n (a_j u + g_j(v))k_j.$$

In this case then we have

$$m\cdot p = 0, \ n\cdot p = 0, \ m\cdot q = 0, \ n\cdot q = 0, \ n\cdot r = 0,$$
$$m\cdot r = \Sigma a_j g_j''.$$

If these values are substituted in (12), we have

$$y_{11} = p, \ y_{12} = 0, \ y_{22} = r - (m\cdot r)m + (m\cdot n)(m\cdot r)n.$$

Hence the curves $u = \text{const.}$ are geodesics on the surface, but in general the curves $v = \text{const.}$ are not. If however $m \cdot r = 0$ both sets of parameter curves are geodesics. Integrating this relation we have

$$\Sigma a_j g_j' = F = \text{const.},$$

and from what we saw previously concerning the relation § 3 this would require that the curve C_2 be a line and that the surface be ruled. Hence if C_1 and C_2 do not lie in completely perpendicular spaces and the translation surface is not ruled, both parameter systems cannot be geodesics on the surface.

The indicatrix for this type of surface does not reduce to a linear segment.

7. *Surfaces for Which C_1 and C_2 Coincide.*—If the curves C_1 and C_2 coincide, the surface (1) becomes the locus of the midpoints of the secants of a fixed curve. The surface is entirely similar to the same case in 3-space, except that in 3-space the fixed curve is an asymptotic line on the surface. Here the curve lies on the surface and is the locus of points at which the characteristics coincide. This curve has the property of an asymptotic line on a surface in 3-space.* The osculating plane of the curve is tangent to the surface and the tangent lines to this curve have three-point contact with the surface.† This is the only such line on the surface.

8. *Minimum surfaces.*—If we use the minimum curves on a surface as parameter lines, the coefficients in the first fundamental form are

$$a_{11} = m \cdot m = 0, \qquad a_{22} = n \cdot n = 0, \qquad a_{12} = m \cdot n.$$

The element of arc then becomes

$$ds^2 = a_{11}du^2 + 2a_{12}dudv + a_{22}dv^2 = 2a_{12}dudv.$$

The general formula for the mean curvature in terms of the covariant derivatives is

$$2h = \Sigma a^{(rs)} y_{rs},$$

where $a^{(rs)}$ is the complement of a_{rs} in the determinant $|a_{ij}|$

* Segre, loc. cit.

† C. L. Moore, "Surfaces in hyperspace, etc.," BULLETIN, vol. 18 (1912).

divided by a. In this case then

$$a^{(11)} = 0, \qquad a^{(22)} = 0, \qquad a^{(12)} = \frac{1}{a_{12}}$$

and the formula for h becomes

$$2h = \frac{1}{a_{12}} y_{12}.$$

The vanishing of the vector mean curvature is the necessary and sufficient condition for a minimum surface. Then, if the surface is minimum,

$$y_{12} = 0.$$

In 3-space if the minimum lines are taken as the parameter curves on a surface, the condition that the surface be minimum is the vanishing of the second coefficient in the second fundamental form. In hyperspace we have the same condition with respect to the vector second fundamental form. From (12) this condition becomes

$$y_{12} = (m \times n) \cdot (m \times n \times q) = 0.$$

The dot product is that used by Wilson and Lewis* and the vanishing here requires that q lie in the plane $m \times n$, that is

$$m \times n \times q = 0.$$

This is equivalent to saying that the coordinates must satisfy the differential equation

$$A \frac{\partial^2 Z}{\partial u \partial v} + B \frac{\partial Z}{\partial u} + C \frac{\partial Z}{\partial v} = 0, \tag{14}$$

which again amounts to saying that there is a linear relation connecting m, n, q

$$Aq + Bm + Cn = 0. \tag{15}$$

Differentiating the relations $m^2 = 0$, $n^2 = 0$, we see that

$$m \cdot q = 0, \qquad n \cdot q = 0.$$

* "Space time manifold of relativity," *Proceedings Amer. Acad.*, vol. 48 (1912).

Multiplying (15) by m, we have

$$Cm \cdot n = Ca_{12} = 0.$$

Since $a_{12} \neq 0$, $C = 0$. Likewise multiplying by n we see that $B = 0$. Hence equation (14) becomes

$$A \frac{\partial^2 Z}{\partial u \partial v} = 0.$$

Hence, *the minimum surface is a surface of translation. The necessary and sufficient condition that a surface in hyperspace be a minimum surface is that the minimum lines on it are characteristics.*

MASSACHUSETTS INSTITUTE OF TECHNOLOGY.

SOME ALGEBRAIC CURVES.

BY DR. JAMES H. WEAVER.

(Read before the American Mathematical Socicty, April 28, 1917.)

IN the following paper two algebraic curves are set up and some of their singularities are discussed. The author believes them to be new. At least a search through considerable of the literature on curves has failed to reveal them.

I.

Let there be any two distinct points A and B. Let the line joining A and B be drawn, and let the distance $AB = c$. Let there be drawn through A a line l_1 making an angle θ with AB, and let there be drawn through B a line l_2 making an angle $n\theta$ with AB (n an integer). We also consider that AB, l_1, and l_2 are in one plane. Let the intersection of l_1 and l_2 be C. It is required to find the locus of C.

Let A be the origin and let AB be the x-axis. Then the equations of the lines l_1 and l_2 will be

$$(1)\ y = x \tan \theta, \qquad (2)\ y = (x - c) \tan (n\theta)$$

respectively.

After eliminating θ from (1) and (2) we get

$$(3)\quad \begin{aligned} &x^n - \binom{n}{2}x^{n-2}y^2 + \binom{n}{4}x^{n-4}y^4 - \cdots + (-1)^{n-1/2}xy^{n-1} \\ &= (x-c)\left[\binom{n}{1}x^{n-1} - \binom{n}{3}x^{n-3}y^2 + \cdots + (-1)^{n-1/2}y^{n-1}\right], \end{aligned}$$

where n is of the form $2k+1$. A similar form holds if n is of the form $2k$. The theory is the same in either case. (3) is then the equation representing the locus of C. Let us call this curve C_n. If in (2) we replace n by $n-r$, we obtain for (3) a curve of degree $n-r$, which we will call C_{n-r}. ($r = 1, 2, \ldots, n-1$.)

From (3) it is evident that C_n has an $(n-1)$-point at the origin. The equations of the $n-1$ tangents at this point will be given by

$$(4)\quad \binom{n}{1}x^{n-1} - \binom{n}{3}x^{n-3}y^2 + \cdots + (-1)^{(n-1/2)}y^{n-1} = 0.$$

The factors of (4) are

$$y - x\tan(k\pi/n) \quad (k = 1, \cdots, n-1).$$

Therefore the tangents to C_n at the point A together with the x-axis divide the angular magnitude about A into $2n$ equal parts.

We will now consider the relation of the fixed point B to the curve C_n. Let us write (3) in homogeneous coordinates. It will then be

$$(5)\quad \begin{aligned} &x^n - \binom{n}{2}x^{n-2}y^2 + \cdots + (-1)^{n-1/2}xy^{n-1} \\ &= (x-cz)\left[\binom{n}{1}x^{n-1} - \cdots + (-1)^{n-1/2}y^{n-1}\right]. \end{aligned}$$

The first polar of this curve with respect to the point $B = (c, 0, 1)$ is

$$(6)\quad \begin{aligned} &x^{n-1} - \binom{n-1}{2}x^{n-3}y^2 - \cdots + (-1)^{n-1/2}y^{n-1} \\ &= (x-cz)\left[\binom{n-1}{1}x^{n-2} - \cdots + (-1)^{n-2/2}y^{n-2}\right]. \end{aligned}$$

This process may evidently be continued. We may then state the following

Theorem: The rth polar of B with respect to C_n is C_{n-r}.

II.

Again let there be three distinct points A, B, and C on the same straight line l, and through the point C let the line l_1 be drawn perpendicular to l. Let lines l_2 and l_3 be drawn through A and B respectively, and let l_2 and l_3 intersect on l_1. Let l_2 make an angle α with l, and l_3 make an angle β with l, and let a line l_4 be drawn through B, making an angle $n\beta$ with l. Let l_2 and l_4 intersect in D. Then just as in section I, the equation representing the locus of D is

$$k\left[x^n - \binom{n}{2}x^{n-2}y^2 + \cdots\right] = (x-c)\left[\binom{n}{1}x^{n-1} - \binom{n}{3}x^{n-3}y^2 + \cdots\right], \tag{7}$$

where $k = (a-c)/a$ and $a = AC$, and $c = AB$.

It is then evident that the theorem in section I holds for the curve represented by equation(7).

OHIO STATE UNIVERSITY.

ON THE RECTIFIABILITY OF A TWISTED CUBIC.

BY DR. MARY F. CURTIS

(Read before the American Mathematical Society, April 27, 1918.)

GIVEN the twisted cubic

$$x_1 = at, \quad x_2 = bt^2, \quad x_3 = ct^3, \quad abc \neq 0; \tag{1}$$

to show that the condition that it is a helix is precisely the condition that it is algebraically rectifiable.

If (1) is a helix, then T/R, the ratio of curvature to torsion, is constant. Denoting differentiation with respect to t by

primes, we have

$$\begin{aligned} x': &\quad a \quad 2bt \quad 3ct^2, \\ x'': &\quad 0 \quad 2b \quad 6ct, \\ x''': &\quad 0 \quad 0 \quad 6c, \end{aligned}$$

$$(x' \mid x') = a^2 + 4b^2t^2 + 9c^2t^4, \quad (x'' \mid x'') = 4(b^2 + 9c^2t^2),$$

$$(x' \mid x'') = 2t(2b^2 + 9c^2t^2), \quad |x'x''x'''| = 12abc,$$

$$(x'x'' \mid x'x'') = (x' \mid x')(x'' \mid x'') - (x' \mid x'')^2$$

$$= 4(a^2b^2 + 9a^2c^2t^2 + 9b^2c^2t^4).$$

$$\frac{T}{R} = -\left(\frac{x'x'' \mid x'x''}{x' \mid x'}\right)^{3/2} \frac{1}{|x'x''x'''|}.$$

Since $|x'x''x'''|$ is constant, T/R is constant when and only when $(x'x'' \mid x'x'')/(x' \mid x')$ is constant. We thus have

$$4(a^2b^2 + 9a^2c^2t^2 + 9b^2c^2t^4) \equiv \rho(a^2 + 4b^2t^2 + 9c^2t^4);$$

hence $\rho = 4b^2$ and $9a^2c^2 - 4b^4 = 0$. Conversely, for all values of a,b,c, $abc \neq 0$, for which

$$9a^2c^2 - 4b^4 = 0, \tag{2}$$

T/R is constant—in particular, is equal to ∓ 1, according as $2b^2 = \pm 3ac$—and the cubic (1) is a helix.

If we had fixed our attention on another characteristic property of a helix, namely, that the tangent makes with a fixed direction a constant angle, we should have again derived the condition (2). The fixed direction—that of the axis of the cylinder on which the helix lies—is $(1/\sqrt{2},\ 0,\ \pm 1/\sqrt{2})$ and the helix cuts the rulings of the cylinder under an angle of 45°.

That (2) is a necessary and sufficient condition that s, the arc of (1), is an algebraic function of t and hence that (1) is algebraically rectifiable follows from the fact that the integral

$$s = \int_{t_0}^{t} \sqrt{a^2 + 4b^2t^2 + 9c^2t^4}\,dt \tag{3}$$

is algebraic when and only when (2) holds. Hence the theorem: The twisted cubic (1) is algebraically rectifiable when and only when it is a helix.

SHORTER NOTICES.

A New Analysis of Plane Geometry Finite and Differential. By A. W. H. THOMPSON. Cambridge, University Press, 1914. 16 + 120 pages. Price 7 shillings.

THIS book is an exposition of a system of notation for plane metric geometry. Points are represented by Latin letters, lines by Greek letters; $\overline{ab}$ denotes the line determined by the points a and b, and $\alpha\beta$ denotes the point determined by the lines α and β; (ab), $(\alpha\beta)$, and $(a\beta)$ denote respectively the distance between a and b, the angle between α and β and the perpendicular distance between a and β. The author's problem is now one of reductional computation—to express in terms of these measures of two elements any measure of elements derived from points and lines by intersections and joins, vectorial constructions, and equational relations. This reduction is carried out not only for such measures as areas of triangles and the trigonometric functions but for differentiation and integration. The author claims that his method is superior to that of coordinate geometry in the matter of sign. There are nearly two hundred examples in the text, the majority of them being grouped at the end.

G. H. GRAVES.

An Elementary Course in Differential Equations. By EDWARD J. MAURUS. Ginn and Company, 1917. vi + 51 pp.

THIS is a collection of about 350 problems covering ordinary differential equations up to solution in series. Very brief explanations of the methods of formation and of solving differential equations are given, and the applications are only hinted at.

C. F. CRAIG.

An Introduction to Statistical Methods. By HORACE SECRIST. New York, Macmillan, 1917. xxi+ 482 pp. $2.00

As stated on the title page, this is "A text book for college students, a manual for statisticians and business executives." The book is descriptive rather than mathematical in character, making its appeal to the general reader through its discussion of methods and purposes.

Sufficient illustrations are given to make the meaning clear. The discussion is detailed, almost wordy at times. This very wordiness makes the book of especial value to the beginner. Each chapter ends with a summary and a list of references to other standard works on the same and allied subjects.

The chapter on averages discusses at some length the arithmetic mean, the mode, and the median and omits the geometric mean, and the mean given by $\sqrt{\Sigma x^2/n}$, and combinations of these averages.

Illustrative matter is mainly from the economic field. A few illustrations are from agriculture. References are made, however, to other fields, such as biology, psychology, genetics.

Although non-mathematical, the book is of interest to a mathematician from the point of view of the applications. This is a good book for a beginner and at the same time useful to one already initiated into the study of statistics.

W. V. LOVITT.

Cours de Mécanique. Vol. III. By LÉON LECORNU. Paris, Gauthier-Villars, 1918. 669 pp.

THIS is the third and last volume of a treatise on mechanics for use in l'Ecole Polytechnique. The first was reviewed in the BULLETIN for April, 1915, and the second, November, 1917.

The present volume is devoted to applications of mechanics to engineering and consists of five parts (parts X to XIV of the complete course). The subjects considered are strength of materials, hydraulics, thermodynamics, theory of machines, and a brief discussion of the problems involved in aviation.

It is interesting to compare the preface to the third volume with the preface to the first, which appeared in 1914. Just before the war, Professor Lecornu, in speaking of the course in mechanics as a whole, referred to the necessity for resisting a demand for the teaching of practical applications and expressed his firm belief that the course in l'Ecole Polytechnique should be purely theoretical. In support of this position he quoted General Langlois. "The officers who leave the school at the end of one year are, in general, inferior to their comrades in the matter of studying logically and deeply a scientific question of tactics or organization. The method of work indispensable to every man of action demands imperiously the study of a science to its foundations, a study which makes

the intellect supple and develops a habit of logical deduction necessary to one who commands."

In the preface to the third volume, written after three and a half years of war, the author explains that the experiences through which the country is passing have influenced him to extend the treatment of applied mechanics beyond the course as it is now given. He foresees, after peace is reestablished, a profound transformation in scientific study, which will adapt it more directly to the realities of life (réalités de la vie). He refers to the action of L'Académie des Sciences in deciding (January, 1918) to admit a certain number of representatives of industry and predicts that l'Ecole Polytechnique will reduce the time now given to abstract theory and increase the time allotted to applied work.

W. R. LONGLEY.

Electric and Magnetic Measurements. By CHARLES MARQUIS SMITH. (Edited by E. R. Hedrick.) New York, Macmillan, 1917. xii + 373 pp.

FROM the point of view of the mathematical reader the interest of this book is purely incidental. Moreover, it is incidental to study not in any general field but only in the special field of the mathematical theory of electricity and magnetism. In fact, the book belongs to a series of texts on topics in engineering. As such a detailed review of it is out of place in this BULLETIN. It is well, however, at this time when more interest is being manifested in applied mathematics than heretofore in this country to have attention directed to a convenient description of the instruments and methods by means of which are measured the quantities involved in the theory of electricity and magnetism, a subject which has lent itself in a remarkable way to precise mathematical treatment. The book under consideration furnishes in convenient form what such a mathematical reader will desire. The fact that it was written for engineering students does not interfere with this use of the book.

R. D. CARMICHAEL.

NOTES.

On account of war conditions the Southwestern Section of the American Mathematical Society will not hold its meeting this year. No eastern meeting will be held until April 26, 1919. The Chicago meeting in the Christmas holidays will be the annual meeting of the Society for the election of officers and other members of the Council, and will be especially marked by the retiring address of President L. E. Dickson.

The third summer meeting of the Mathematical Association of America was held at Dartmouth College on Thursday, Friday, and Saturday, September 5–7, 1918, immediately following the summer meeting of the American Mathematical Society at the same place. A joint dinner of the two organizations, on Thursday evening, was attended by fifty-six members and friends. At the joint session on Friday morning Professor A. G. Webster gave an address on "Mathematics of warfare." The retiring address of President Cajori: "Plans for a history of mathematics of the nineteenth century," was read by Dean G. D. Olds. The other papers on the regular programme were: "The teaching of curve tracing," by F. W. Owens; "A formula in combinatorial analysis," by J. W. Young; "Trigonometric functions—of what?," by W. B. Carver; "Firing data at Yale," by J. K. Whittemore; "A combined course in mathematics for college freshmen," by A. S. Gale; Report of the committee on mathematical requirements, by J. W. Young, chairman; "Some experiments in the teaching of descriptive geometry," by F. L. Kennedy. A very great interest was shown in the new courses for military training in the colleges, and nearly all the members present cooperated actively with a committee appointed to prepare a draft of such courses for submission to the authorities. The total attendance at the sessions numbered seventy-one, including forty-two members. Fourteen new members were elected.

The July number of the *Transactions of the American Mathematical Society* contains the following papers: "Singular points

of analytic transformations," by W. F. OSGOOD; "Space involutions defined by a web of quadrics," by VIRGIL SNYDER and F. R. SHARPE; "On the location of the roots of the Jacobian of two binary forms, and of the derivative of a rational function," by J. L. WALSH; "Sets of independent generators of a substitution group," by G. A. MILLER; "The problem of Mayer with variable end points," G. A. BLISS.

PROFESSORS O. D. KELLOGG, of the University of Missouri, and MAX MASON, of the University of Wisconsin, are engaged in government service at the Naval Experimental Station at New London, Conn.

THE following university teachers are giving instruction in mathematics at the field artillery central officers' training school at Camp Zachary Taylor, Ky.: Dr. L. R. FORD, Harvard University; Mr. R. W. BARNARD and Professor T. H. HILDEBRANDT, University of Michigan; Mr. J. D. ESHLEMAN, University of Rochester; Dr. T. R. HOLLCROFT, Columbia University; Dr. C. E. WILDER and Professor C. H. YEATON, Northwestern University; Mr. H. E. WOLFE, University of Indiana. Mr. R. E. MOORE, University of Wisconsin, died October 2.

PROFESSOR G. N. WATSON, of Trinity College, Cambridge, and University College, London, has been appointed to the professorship of mathematics in the University of Birmingham, in succession to Professor R. S. Heath.

AT the University of Saskatchewan assistant professor L. L. DINES has been promoted to a full professorship of mathematics.

DR. R. M. WINGER, of the University of Oregon, has been appointed assistant professor of mathematics in the University of Washington.

DR. F. W. BEAL, of the University of Pennsylvania, has been appointed professor of mathematics in the University of Tennessee.

DR. L. E. WEAR, of the University of Washington, has been appointed associate professor of mathematics in Throop College.

DR. J. R. KLINE, of the University of Pennsylvania, has been appointed instructor in mathematics in the Sheffield Scientific School of Yale University.

PROFESSOR PAOLO PIZZETTI, of the University of Pisa, died April 14, 1916, at the age of fifty-eight years.

BOOK CATALOGUES: Galloway and Porter, Cambridge, England, catalogue 93, about 100 titles in mathematics and physics.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BIANCHI (L.). Lezioni sulla teoria dei gruppi continui finiti di transformazioni. Pisa, E. Spoerri (F. Mariotti), 1918. 8vo. 6 + 590 pp. L. 25.00

CANDIDO (G.). La risultante di due quadratiche. Livorno, tip. R. Giusti, 1918. 4to. 26 pp.

CHISINI (O.). See ENRIQUES (F.).

COLPITTS (J. T.). See ROBERTS (M. M.).

ENRIQUES (F.). Conferenze sulla geometria non-euclidea, per cura del O. Fernandez. Bologna, Zanichelli, 1918. 8vo. 46 pp. L. 3.00

——. Lezioni sulla teoria geometrica delle equazioni e delle funzioni algebriche, pubblicate per cura O. Chisini. Volume 2. Bologna, Zanichelli, 1918. 8vo. 713 pp. L. 25.00

FERNANDEZ (O.). See ENRIQUES (F.).

GAUSS (C. F.). Werke. Herausgegeben von der Königlichen Gesellschaft der Wissenschaften zu Göttingen. 10ter Band, 1te Abteilung. Leipzig, Teubner, 1917. 4to. 586 pp.

HUYGENS (C.). Du calcul dans les jeux de hasard, rédigé par D. J. Korteweg. La Haye, Martinus Nijhoff, 1917. 4to. 179 pp.

KORTEWEG (D. J.). See HUYGENS (C.).

PASCAL (E.). Lezioni di calcolo infinitesimale. Parte 1: Calcolo differenziale. 4a edizione riveduta. (Manuale Hoepli.) Milano, Hoepli, 1918. 24mo. 12 + 313 pp. L. 4.50

PEZZO (P. DEL). Principî di geometria projettiva: lezioni dettate nell' università di Napoli nell' anno 1917–1918. Napoli, tip. B. De Rubertis, 1918. 8vo. 116 pp.

ROBERTS (M. M.) and COLPITTS (J. T.). Analytic geometry. New York, Wiley, 1918. 10 + 229 pp. $1.60

TORELLI (G.). Lezioni di calcolo infinitesimale date nella r. università di Napoli. Napoli, tip. Accademia r. delle Scienze fisiche e matematiche, 1918. 8vo. 4 + 424 pp. L. 20.00

II. ELEMENTARY MATHEMATICS.

BOSTON, DEPARTMENT OF EDUCATIONAL INVESTIGATION AND MEASUREMENT. Arithmetic; the value to the teacher, to the principal and to the superintendent of individual and class record from standard tests. (Bulletin 13; School documents 22, 1917.) Boston, School Committee, 1917. 8vo. 83 pp.

CLARK (J. R.). See RUGG (H. O.).

CORRELL (F. E.) and FRANCIS (M. E.). Exercises in arithmetic. Waverly, Iowa, Francis-Correll Company, 1917. 8vo. 71 pp. $0.25

DAVIS (R.). Business practice in elementary schools. (Harvard Studies in Education.) Cambridge, Harvard University Press, 1917. 8vo. 31 pp. Paper. $0.30

FRANCIS (M. E.). See CORRELL (F. E.).

IANNELLI (G.). Di un metodo pratico per la trisezione dell' angolo. Palermo, tip. Pontificia, 1918. 4to. 9 pp. + 1 tavola.

——. Studio sulla trisezione dell' angolo. Palermo, tip. Pontificia, 1917. 4to. 11 pp. + 2 tavole.

LINDQUIST (T.). Modern arithmetic, methods and problems. Edited by G. W. Myers. Chicago, Scott, Foresman and Company, 1917. 8vo. 16 + 300 pp. $1.25

MÜLLER (O.). Tavole di logaritmi con cinque decimali. 14a edizione, aumentata delle tavole dei logaritmi d'addizione e sottrazione per cura di M. Rajna. (Manuali Hoepli.) Milano, Hopeli, 1918. 24mo. 36 + 191 pp. L. 1.50

MYERS (G. W.). See LINDQUIST (T.).

OWEN (L.). Work-book in arithmetic; grade four. Boston, Mansfield Printing Company, 1917. $0.30

RAJNA (M.). See MÜLLER (O.).

RUGG (H. O.).and CLARK (J. R.). Scientific method in the reconstruction of ninth-grade mathematics, a complete report of the investigation of the Illinois committee on the standardization of ninth-grade mathematics, 1913–1918. Chicago, University of Chicago Press, 1918. 8vo. 189 pp. Paper. $1.00

SAMPSON (C. H.). Assignment manual of algebra. Boston, Sanborn, 1917. 8vo. 8 + 53 pp. $0.50

III. APPLIED MATHEMATICS.

ANFINDSEN (S.). Altitude correction tables. 2d edition. New York, Rudder Publishing Company, 1917. 8vo. 32 pp. $1.00

ANDREWS (E. S.). A primer of engineering science. Part 2: First step. in heat and heat engines. London, J. Selwyn and Company, 1918. 9 + 67 pp.

BRADFORD (G.). Whys and wherefores of navigation. New York, Van Nostrand, 1918. 8vo. 10 + 163 pp. Leather. $2.00

BROWN (H.T.). 600 meccanismi scelti fra i più importanti e recenti riferentisi alla dinamica, idraulica, idrostatica, pneumatica, macchine-vapore, molini, torchi, automobili, orologeria ed altre diverse macchine. 6a edizione italiana, aumentata di altre 100 nuovi meccanismi, per cura del C. Marvasi. (Manuali Hoepli.) Milano, Hoepli, 1918. 24mo. 8 + 295 pp. L. 4.50

CHINI (M.). Coros speciale di matematiche, con numerose applicazioni, ad uso specialmente dei chimici e dei naturalisti. 3a edizione. Livorno, R. Giusti, 1918. 8vo. 12 + 297 pp. L. 6.50

CONTALDI (P.). La meccanica e le macchine nella scuola e nell' industria. Volume 1. Meccanica generale: statica, cinematica, dinamica. Meccanica applicata: resistenza dei materiali, resistenze passive, transmissioni, meccanica applicata alle macchine. 4a edizione, riveduta ed ampliata. (Biblioteca Tecnica.) Milano, Hoepli, 1918. 8vo. 15 + 678 pp. L. 18.50

CROCCO (A.). See NOBILE (U.).

DINARO (S.). La meccanica industriale nelle scuole e per l'officia: trattato metodico. 2a edizione, rifatta ed ampliata, con appendice sullo stampaggio e lavorazione dei proiettili per le artiglierie, note e proposte sulle scuole operaie prima e dopo la guerra. (Manuali Hoepli.) Milano, Hoepli, 1918. 24mo. 500 pp. L. 6.50

EUCLID. See OVIO (G.).

GIORLI (E.). Meechanista navale e construttore meccanico, per uso dei macchinisti della r. marina, dei macchinisti delle compagnie di navigazione, dei periti e construttori navali-meccanici, capitecnici congregnatori, . . . 2a edizione, completamente rifatta. (Manuali Hoepli.) Milano, Hoepli, 1917. 24mo. 16 + 575 pp. L. 8.50

GUPTON (T. A.). Mathematical geography as outlined in the state course of study; eighth year. Emden, Ill., T. A. Gupton, 1917. $0.25

JACOBY (H.). Navigation. 2d edition with a chapter on compass adjusting and a collection of miscellaneous examples. New York, Macmillan, 1918. 8vo. 11 + 350 pp. $2.25

LONGONI (E.). La navigazione aerea. I: Aerostati e dirigibili. (Biblioteca dei Popoli, Ni. 467–468.) Milano, Casa ed. Sonzogno (Matarelli), 1918. 16mo. 124 pp. L. 0.60

MALAVASI (C.). See MANUALE.

MANUALE enciclopedico della ingegneria moderna, compilato da sessanta ingegneri specialisti, per cura della Società accademica Hütte. Edizione italiana autorizzata, curata sull' ultima (22a) edizione originale. Revisione generale C. Malavasi. Volume 1. Milano, Hoepli (Firenze, tip. l'Arte della Stampa, succ. Laudi), 1918. 16mo. 16 + 1063 pp.

MARCHI (G.). Manuale pratico per l'operaio elettrotecnico. 6a edizione, rinnovata ed ampliata. (Manuale Hoepli.) Milano, Hoepli (tip. Sociale), 1918. 24mo. 12 + 394 pp. L. 6.50

MARVASI (C.). See BROWN (H. T.).

NOBILE (U.). L'aviazione: fondamenti sperimentali e teorici, con prefazione del colonnello A. Crocco. Roma, tip. del Genio civile, 1918. 8vo. 186 pp. + 1 tavola. L. 6.50

OVIO (G.). L'ottica di Euclide. (Manuali Hoepli.) Milano, Hoepli, 1918. 24mo. 20 + 415 pp. L. 7.50

REGNO (W. DEL). Radiotelegrafia, radiotelefonia ed aviazione: prolusione letta al collegio romano per l'inaugurazione della scuola civile di aeronautica il 1° ottobre 1917. Roma, tip. E. Armani, 1918. 16mo. 20pp.

TOLMAN (R. C.). The theory of the relativity of motion. Berkeley, University of California Press, 1917. 10 + 225 pp.